Random dispersion coefficients and Tsallis entropy

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Tsallis Entropy & q-Gaussians

Tsallis entropy [1] for a discrete system:

$$S_q = \frac{1}{q-1}\left(1 - \sum_{i=1} p_i^q\right)$$

Tsallis entropy for a continuous random variable:

$$S_q(X) = \frac{1}{q-1}\left(1 - \int_{-\infty}^{\infty} [f_X(x)]^q \, dx\right)$$

The q-Gaussian distribution,

$$f(x) = \frac{\sqrt{\beta}}{C_q} e_q(-\beta x^2)$$

where

$$e_q(x) = [1 + (1-q)x]^{1/(1-q)}$$

maximizes the Tsallis entropy for a fixed second moment ($\langle X^2 \rangle_q = \sigma^2$) of the q-expectation

$$\langle g(X) \rangle_q \equiv \frac{\int_{-\infty}^{\infty} g(x) [f_X(x)]^q \, dx}{\int_{-\infty}^{\infty} [f_X(x)]^q \, dx},$$

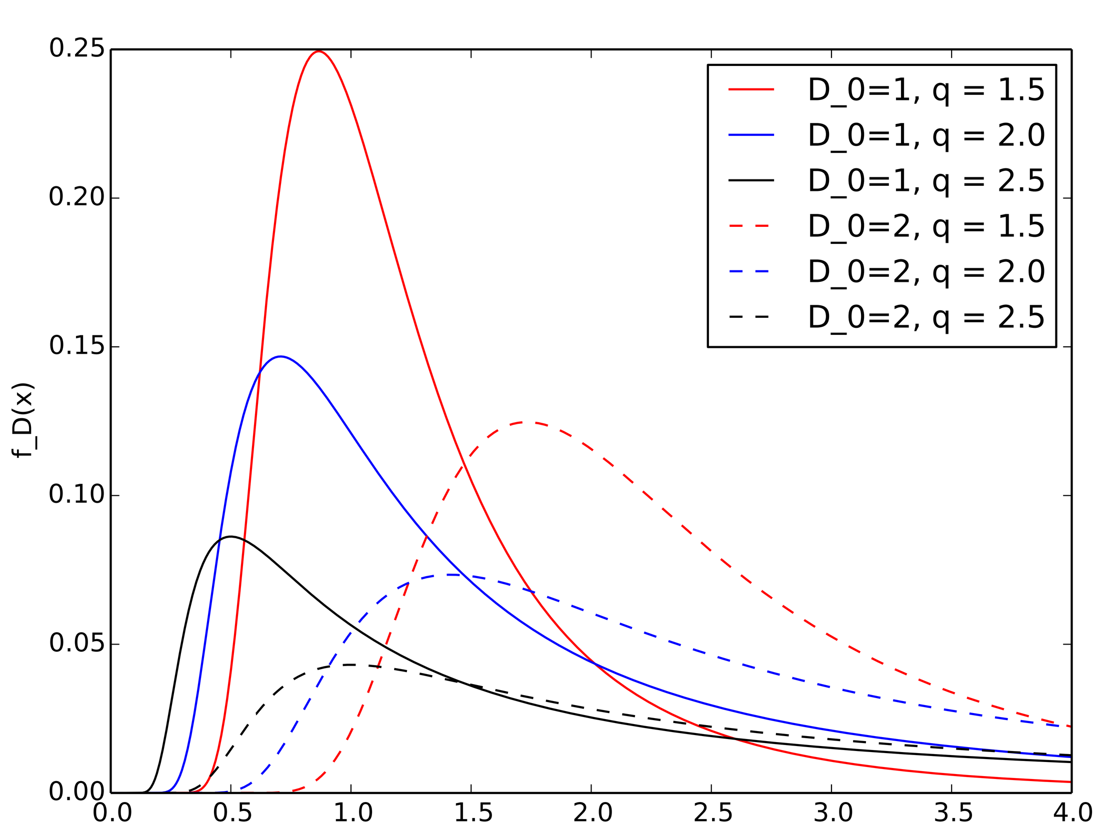
Dispersion Coefficient PDF

Consider the stochastic differential equation

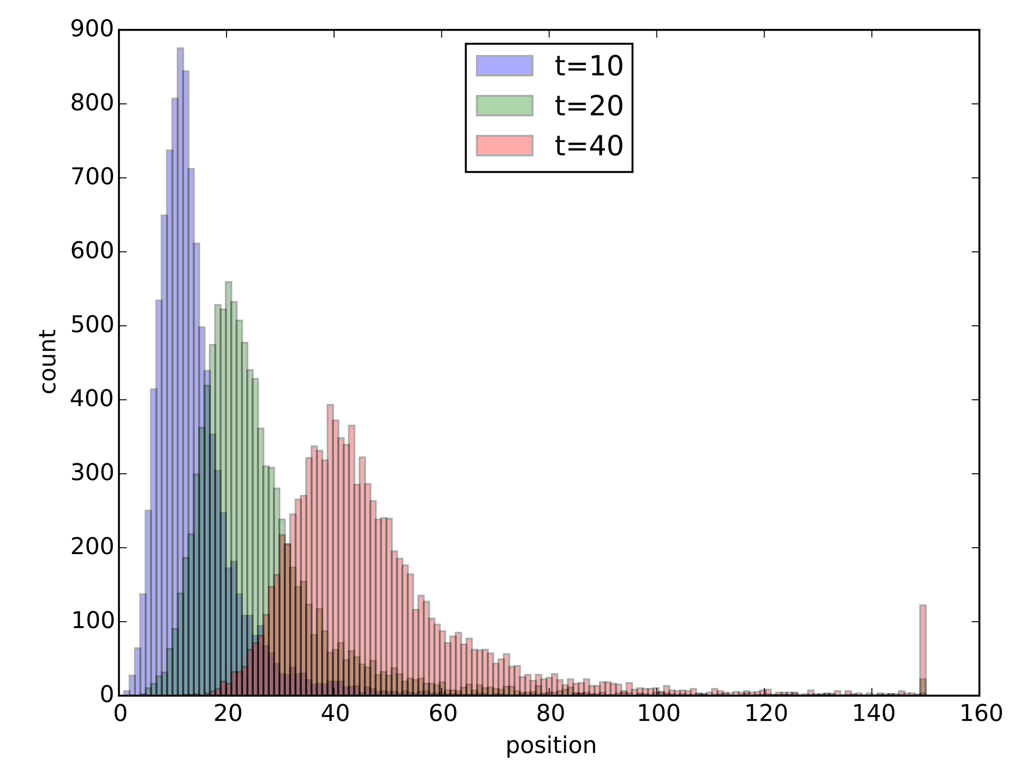
$$dX(t) = v dt + \sqrt{D} dB(t)$$

where $B(t)$ is a Brownian motion, and D is a random variable that is independent of $B(t)$. What distribution of D maximizes the Tsallis entropy of the dispersion? This one ($\nu = \frac{q-3}{1-q}$):

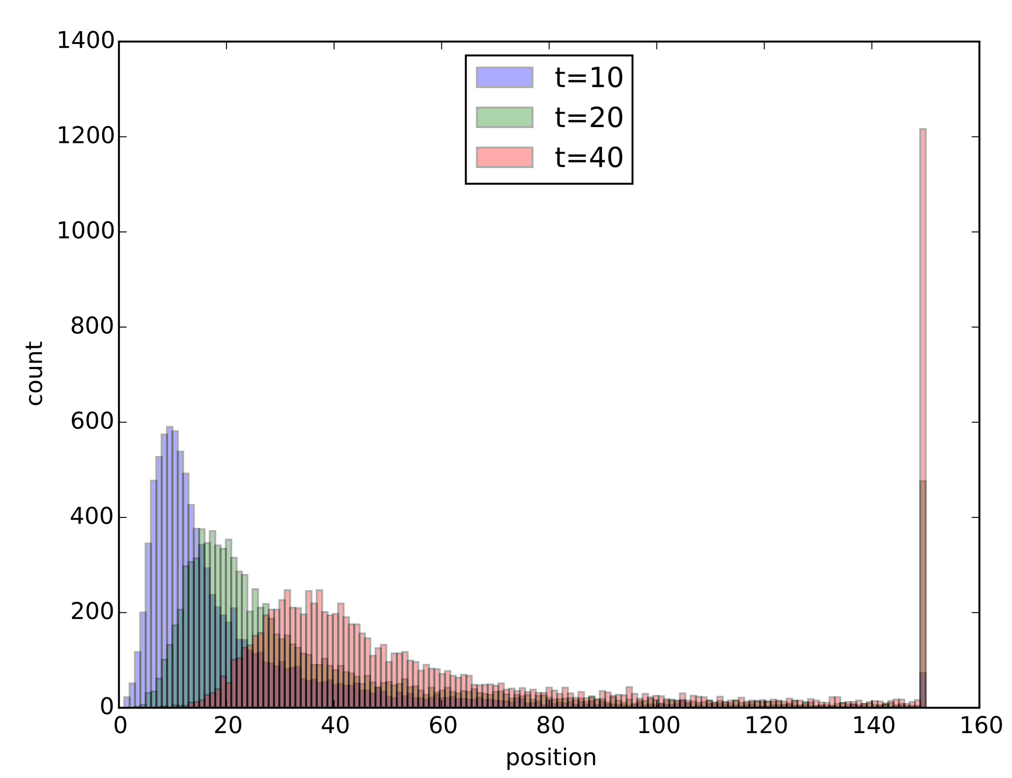
$$f_D(x) = \frac{1}{2^{\nu/2+1}\Gamma(\nu/2)} (D_0^2 \nu)^{\nu/2} x^{-\nu-1} e^{-D_0^2 \nu / 2x^2}$$



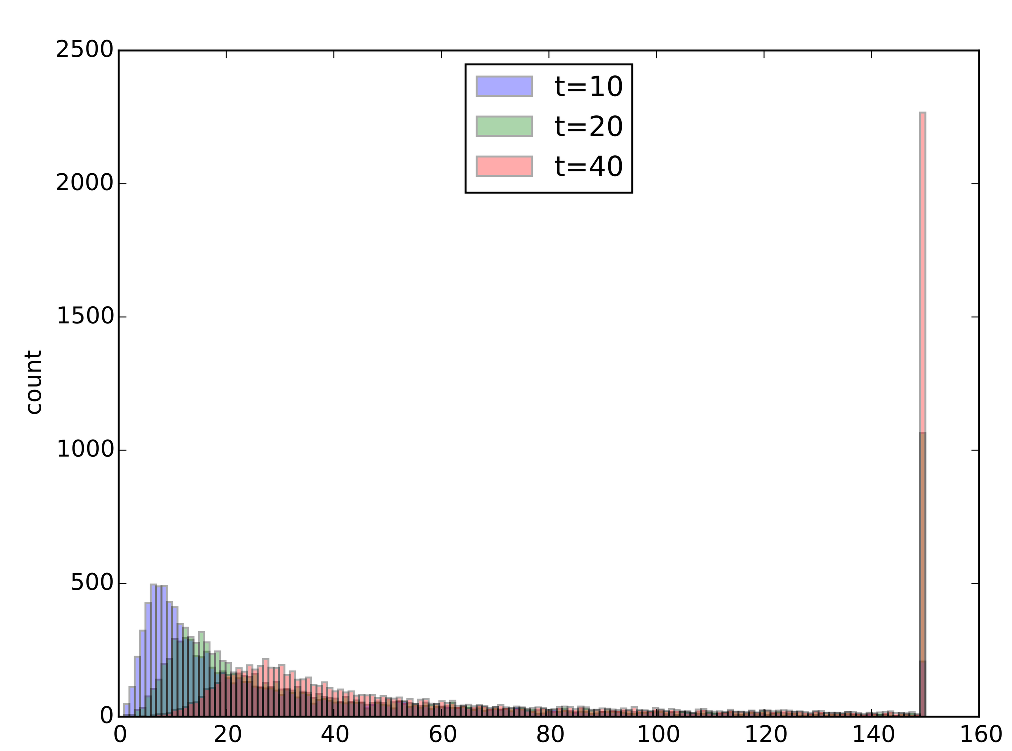
$q = 1.5$ Position Histogram



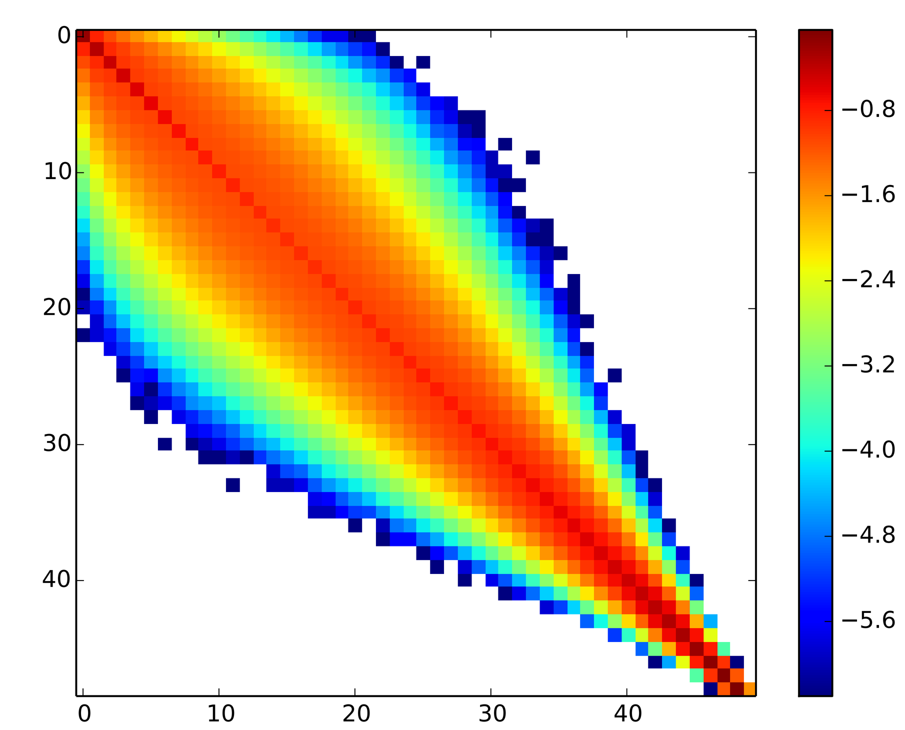
$q = 2$ Position Histogram



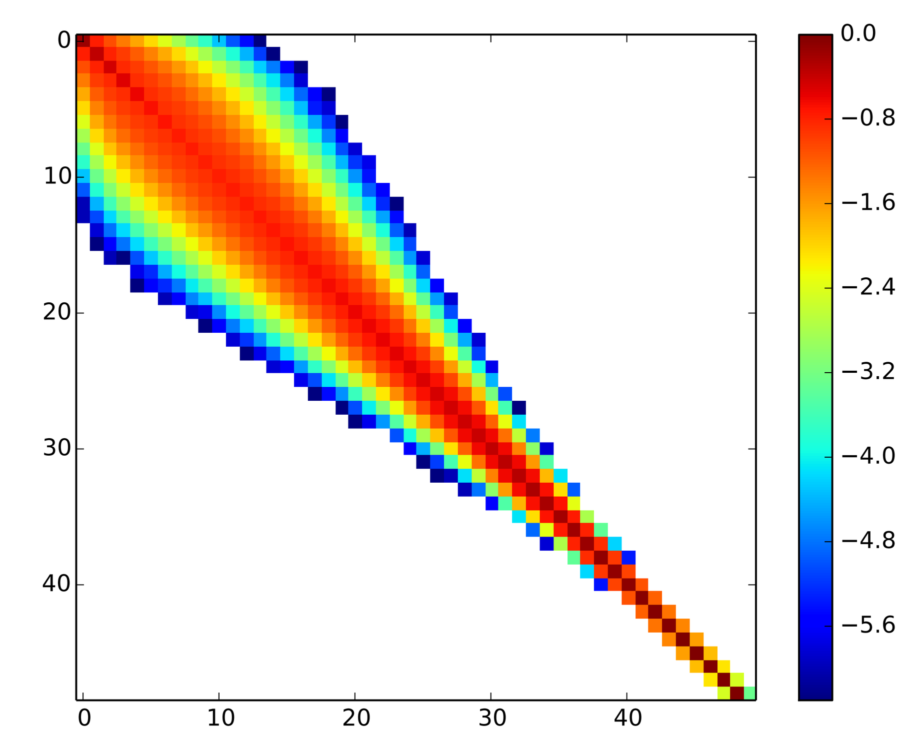
$q = 2.5$ Position Histogram



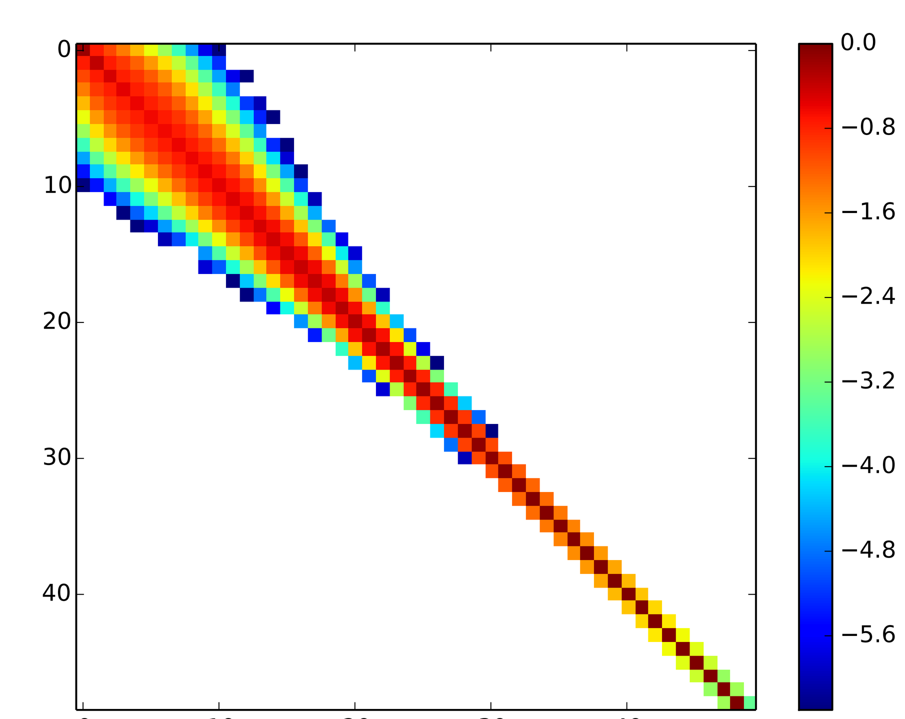
$q = 1.5$ Velocity Transitions



$q = 2$ Velocity Transitions



$q = 2.5$ Velocity Transitions



Properties of Entropy

The Boltzmann-Gibbs entropy results from the following four properties.

1. Entropy is continuous with respect to the probability distribution of states.
2. Entropy is maximal for the uniform distribution.
3. Adding a state with zero probability does not alter the entropy.
4. ~~The entropy of a joint system $A+B$ (where $A+B$ denotes the system obtained by joining the disjoint systems A and B) is the entropy of A plus the expected value of the entropy of B conditioned on A.~~

By dropping the physically dubious 4^{th} property, we obtain a broader set of entropies that includes the Tsallis entropy.

Conclusions

- Gaussian plumes are rarely (or never) observed in natural porous media, therefore dispersion is not maximizing the Boltzmann-Gibbs entropy
- We should look to maximize alternative forms of entropy
- A random dispersion coefficient can be used to maximize Tsallis entropy
- The random dispersion coefficient can be used to inform spatially Markovian models [3] (see figures to the left)
- The random dispersion coefficient provides a maximum-entropy motivation for Lévy dispersion [4] in porous media

References

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3. Le Borgne, T., M. Dentz, and J. Carrera. *Lagrangian statistical model for transport in highly heterogeneous velocity fields.* PRL (2008).
4. Benson, D.A., R. Schumer, M.M. Meerschaert, and S.W. Wheatcraft. *Fractional dispersion, Lévy motion, and the MADE tracer tests.* TiPM (2001).